

# Oscillation of Second Order Nonlinear Difference Equation with Continuous Variable<sup>1</sup>

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*Submitted by William F. Ames*

Received January 28, 2000

In this paper, we are mainly concerned with the second order nonlinear difference equation with continuous variable. Here, by using the iterated integral transformations, generalized Riccati transformations, and integrating factors, we give some oscillatory criteria for this equation. © 2001 Academic Press

*Key Words:* continuous variable; oscillation; integral transformations; generalized Riccati transformations.

## 1. INTRODUCTION

We consider the second order nonlinear difference equation with continuous variable

$$\Delta_{\tau}^2 x(t) + f(t, x(t - \sigma)) = 0, \quad (1.1)$$

<sup>1</sup>Research was supported by the National Natural Science Foundation of China and the Natural Science Foundation of Hebei Province.

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where  $\Delta_\tau x(t) = x(t + \tau) - x(t)$ ,  $\tau, \sigma > 0$ ,  $f \in C([t_0, \infty)) \times R, R)$ ,  $\frac{f(t, u)}{u} \geq p(t) > 0$  for  $u \neq 0$ ,  $p \in C(R, R^+)$ .

A function  $x(t)$  is called the solution of (1.1) if  $x(t) = \varphi(t)$ ,  $\varphi \in C([t_0 - \max\{\tau, \sigma\}, t_0], R)$  and it satisfies (1.1) when  $t \geq t_0$ .

A solution  $x(t)$  is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

Recently, there has been an increasing interest in the study of the oscillation of the first order difference equations with continuous variable such as [1–5]. In this paper, we are mainly concerned with the second order nonlinear difference equation with continuous variable. By using the iterated integral transformations, generalized Riccati transformations, and integrating factors, we give some oscillatory criteria for (1.1).

## 2. MAIN RESULTS

**THEOREM 1.** Assume that  $q(t) = \min_{t \leq s \leq t+2\tau} \{p(s)\}$  and for any  $t \geq t_0$ , there exists a  $t' \geq t$  such that

$$\sum_{i=0}^{\infty} q(t' + i\tau) = \infty. \quad (2.1)$$

Then every solution of (1.1) is oscillatory.

*Proof.* Suppose to the contrary. Let  $x(t)$  be an eventually positive solution of (1.1). Set

$$u(t) = \int_t^{t+\tau} ds \int_s^{s+\tau} x(\theta) d\theta.$$

Thus  $u(t) > 0$ ,  $u''(t) = \Delta_\tau^2 x(t) \leq 0$ ,  $u'(t) > 0$  eventually. Then (1.1) becomes

$$\begin{aligned} u''(t) + f(t, x(t - \sigma)) &= 0, \\ u''(t) + p(t)x(t - \sigma) &\leq 0. \end{aligned} \quad (2.2)$$

Integrating (2.2) in the region

$$D = \begin{cases} t \leq s \leq t + \tau, \\ s \leq u \leq s + \tau, \end{cases}$$

we have

$$\Delta_\tau^2 u(t) + q(t)u(t - \sigma) \leq 0. \quad (2.3)$$

Define  $z(t) = \Delta_\tau u(t)/u(t - \sigma)$ . Since  $u'(t) > 0$ , it follows that  $\Delta_\tau u(t) = u(t + \tau) - u(t) > 0$ ,  $z(t) > 0$ . Then

$$\begin{aligned}\Delta_\tau z(t) &= \frac{u(t + \tau - \sigma)\Delta_\tau^2 u(t) - \Delta_\tau u(t + \tau)\Delta_\tau u(t - \sigma)}{u(t + \tau - \sigma)u(t - \sigma)} \\ &\leq -q(t) - z^2(t + \tau) \leq 0.\end{aligned}$$

Therefore

$$\Delta_\tau z(t + i\tau) + q(t + i\tau) + z^2(t + (i + 1)\tau) \leq 0, \quad i = 0, 1, 2, \dots \quad (2.4)$$

Summing (2.4) from 0 to  $k - 1$

$$z(t + k\tau) - z(t) + \sum_{i=0}^{k-1} q(t + i\tau) + \sum_{i=0}^{k-1} z^2(t + (i + 1)\tau) \leq 0.$$

Hence we have

$$\sum_{i=0}^{k-1} q(t + i\tau) < z(t).$$

Then

$$\sum_{i=0}^{\infty} q(t + i\tau) \leq z(t) < \infty,$$

which contradicts (2.1). This completes the proof.

**THEOREM 2.** Assume that there exists a positive continuous function  $A(t)$  with  $\Delta_\tau A(t) \leq 0$  and for any  $t \geq t_0$ , there exists a  $t' \geq t$  such that

$$\begin{aligned}\sum_{i=0}^{\infty} A(t' + i\tau) \left\{ q(t' + i\tau) + \left( \frac{\Delta_\tau A(t' + i\tau)}{2A(t' + i\tau)} \right)^2 \right. \\ \left. + \Delta_\tau \left( \frac{\Delta_\tau A(t' + (i - 1)\tau)}{2A(t' + (i - 1)\tau)} \right) \right\} = \infty.\end{aligned} \quad (2.5)$$

Then every solution of (1.1) is oscillatory.

*Proof.* Suppose to the contrary. Let  $x(t)$  be a eventually positive solution of (1.1). From the proof of Theorem 1, there exists  $u(t) > 0$  such that  $\Delta_\tau u(t) > 0$  and

$$\Delta_\tau^2 u(t) + q(t)u(t - \sigma) \leq 0.$$

Define

$$z(t) = A(t) \left\{ \frac{\Delta_\tau u(t)}{u(t - \sigma)} - \frac{\Delta_\tau A(t - \tau)}{2A(t - \tau)} \right\}.$$

Then  $z(t) > 0$  and

$$\begin{aligned}
 \Delta_\tau z(t) &= \frac{\Delta_\tau A(t)}{A(t+\tau)} z(t+\tau) \\
 &\quad + A(t) \left\{ \frac{u(t+\tau-\sigma) \Delta_\tau^2 u(t) - \Delta_\tau u(t+\tau) \Delta_\tau u(t-\sigma)}{u(t+\tau-\sigma) u(t-\sigma)} \right. \\
 &\quad \left. - \Delta_\tau \left( \frac{\Delta_\tau A(t-\tau)}{2A(t-\tau)} \right) \right\} \\
 &\leq \frac{\Delta_\tau A(t)}{A(t+\tau)} z(t+\tau) - A(t) \left\{ \left( \frac{\Delta_\tau u(t+\tau)}{u(t+\tau-\sigma)} \right)^2 + q(t) \right. \\
 &\quad \left. + \Delta_\tau \left( \frac{\Delta_\tau A(t-\tau)}{2A(t-\tau)} \right) \right\} \\
 &= -A(t) \left\{ q(t) + \left( \frac{\Delta_\tau A(t)}{2A(t)} \right)^2 + \Delta_\tau \left( \frac{\Delta_\tau A(t-\tau)}{2A(t-\tau)} \right) \right\} - \frac{A(t) z^2(t+\tau)}{A^2(t+\tau)}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \Delta_\tau z(t+i\tau) + A(t+i\tau) &\left\{ q(t+i\tau) + \left( \frac{\Delta_\tau A(t+i\tau)}{2A(t+i\tau)} \right)^2 \right. \\
 &\left. + \Delta_\tau \left( \frac{\Delta_\tau A(t+(i-1)\tau)}{2A(t+(i-1)\tau)} \right) \right\} + \frac{A(t+i\tau) z^2(t+(i+1)\tau)}{A^2(t+(i+1)\tau)} \leq 0. \quad (2.6)
 \end{aligned}$$

Summing (2.6) from 0 to  $k-1$ , we have

$$\begin{aligned}
 z(t+k\tau) - z(t) + \sum_{i=0}^{k-1} A(t+i\tau) &\left\{ q(t+i\tau) + \left( \frac{\Delta_\tau A(t+i\tau)}{2A(t+i\tau)} \right)^2 \right. \\
 &\left. + \Delta_\tau \left( \frac{\Delta_\tau A(t+(i-1)\tau)}{2A(t+(i-1)\tau)} \right) \right\} + \sum_{i=0}^{k-1} \frac{A(t+i\tau) z^2(t+(i+1)\tau)}{A^2(t+(i+1)\tau)} \leq 0.
 \end{aligned}$$

Thus we get

$$\begin{aligned}
 \sum_{i=0}^{\infty} A(t+i\tau) &\left\{ q(t+i\tau) + \left( \frac{\Delta_\tau A(t+i\tau)}{2A(t+i\tau)} \right)^2 \right. \\
 &\left. + \Delta_\tau \left( \frac{\Delta_\tau A(t+(i-1)\tau)}{2A(t+(i-1)\tau)} \right) \right\} \leq z(t) < \infty,
 \end{aligned}$$

which contradicts (2.5). The proof is completed.

*Remark 1.* If (2.1) or (2.5) does not hold, by using similar methods in the papers [6, 7], we can obtain some *meticulous conditions* for the oscillation.

THEOREM 3. Assume that  $\bar{q}(t) = \min_{t \leq s \leq t+2\tau} \{p(s)/\tau^2\}$ ,

$$\int_t^\infty \bar{q}(s) ds = \infty. \quad (2.7)$$

Then every solution of (1.1) is oscillatory.

*Proof.* Suppose to the contrary. From the proof of Theorem 1, there exists  $u(t) > 0$  such that  $u'(t) > 0$  and

$$\Delta_\tau^2 u(t) + \tau^2 \bar{q}(t) u(t - \sigma) \leq 0. \quad (2.8)$$

Define

$$v(t) = \int_t^{t+\tau} ds \int_s^{s+\tau} u(\theta) d\theta > 0.$$

Since  $u'(t) > 0$ , we have  $v(t) \leq \tau^2 u(t + 2\tau)$ ,  $v(t - \sigma - 2\tau) \leq \tau^2 u(t - \sigma)$ , and  $v''(t) = \Delta_\tau^2 u(t) \leq 0$ . Thus  $v'(t) > 0$ . Therefore (2.8) becomes

$$v''(t) + \bar{q}(t) v(t - \sigma - 2\tau) \leq 0.$$

Let  $w(t) = v'(t)/v(t - \sigma - 2\tau)$ . Then

$$\begin{aligned} w'(t) &= \frac{v(t - 2\tau - \sigma)v''(t) - v'(t)v'(t - \sigma - 2\tau)}{v^2(t - 2\tau - \sigma)} \\ &\leq -\bar{q}(t) - \frac{v'(t)v'(t - \sigma - 2\tau)}{v^2(t - 2\tau - \sigma)}. \end{aligned}$$

Since  $v''(t) \leq 0$ ,  $v'(t) > 0$ , it follows that

$$w'(t) \leq -\bar{q}(t) - \left( \frac{v'(t)}{v(t - 2\tau - \sigma)} \right)^2 \leq -\bar{q}(t) - w^2(t). \quad (2.9)$$

Integrating (2.9) from  $t$  to  $T$ , we have

$$w(T) - w(t) \leq - \int_t^T \bar{q}(s) ds - \int_t^T w^2(s) ds.$$

Thus we get

$$\begin{aligned} \int_t^T \bar{q}(s) ds &\leq w(t), \\ \int_t^\infty \bar{q}(s) ds &\leq w(s) < \infty, \end{aligned}$$

which contradicts (2.7). This completes the proof.

**THEOREM 4.** Assume that there exists a function  $A(t)$  with  $A \in C'([t_0, \infty], (0, \infty))$  such that  $A'(t) \leq 0$  and

$$\int_t^\infty A(s) \left\{ \bar{q}(s) + \frac{(A'(s))^2}{4(A(s))^2} + \left( \frac{A'(s)}{2A(s)} \right)' \right\} ds = \infty. \quad (2.10)$$

Then every solution of (1.1) is oscillatory.

*Proof.* Suppose to the contrary. By the proof of Theorem 3, there exists a function  $v(t)$  such that  $v'(t) > 0$  and

$$v''(t) + \bar{q}(t)v(t - \sigma - 2\tau) \leq 0.$$

Let  $w(t) = A(t)\{v'(t)/v(t - 2\tau - \sigma) - A'(t)/(2A(t))\} > 0$ . Then

$$\begin{aligned} w'(t) &= A'(t) \left\{ \frac{v'(t)}{v(t - 2\tau - \sigma)} - \frac{A'(t)}{2A(t)} \right\} \\ &\quad + A(t) \left\{ \frac{v''(t)v(t - 2\tau - \sigma) - v'(t)v'(t - 2\tau - \sigma)}{v^2(t - 2\tau - \sigma)} - \left( \frac{A'(t)}{2A(t)} \right)' \right\} \\ &\leq \frac{A'(t)}{A(t)} w(t) - A(t) \left\{ \bar{q}(t) + \left( \frac{A'(t)}{2A(t)} \right)' + \left( \frac{v'(t)}{v(t - \sigma - 2\tau)} \right)^2 \right\} \\ &= \frac{A'(t)}{A(t)} w(t) - A(t) \left\{ \bar{q}(t) + \left( \frac{A'(t)}{2A(t)} + \frac{w(t)}{A(t)} \right)^2 + \left( \frac{A'(t)}{2A(t)} \right)' \right\} \\ &= -A(t) \left\{ \bar{q}(t) + \left( \frac{A'(t)}{2A(t)} \right)^2 + \left( \frac{A'(t)}{2A(t)} \right)' \right\} - \frac{w^2(t)}{A(t)}. \end{aligned}$$

Thus

$$\begin{aligned} w(T) - w(t) &\leq - \int_t^T A(s) \left\{ \bar{q}(s) + \left( \frac{A'(s)}{2A(s)} \right)^2 + \left( \frac{A'(s)}{2A(s)} \right)' \right\} ds \\ &\quad - \int_t^T \frac{w^2(s)}{A(s)} ds. \end{aligned}$$

So

$$\int_t^\infty A(s) \left\{ \bar{q}(s) + \left( \frac{A'(s)}{2A(s)} \right)^2 + \left( \frac{A'(s)}{2A(s)} \right)' \right\} ds \leq w(t) < \infty,$$

which contradicts (2.10). The proof is completed.

**THEOREM 5.** In addition to the assumptions of Theorem 4, further assume that

$$0 < Q_1(t) = \int_t^\infty A(s) \left\{ \bar{q}(s) + \left( \frac{A'(s)}{2A(s)} \right)^2 + \left( \frac{A'(s)}{2A(s)} \right)' \right\} ds < \infty, \quad (2.11)$$

but

$$\int_t^\infty \frac{Q_1^2(s)}{A(s)} \exp\left(2 \int_t^s \frac{Q_1(\theta)}{A(\theta)} d\theta\right) ds = \infty. \quad (2.12)$$

Then every solution of (1.1) is oscillatory.

*Proof.* Suppose to the contrary. From the proof of Theorem 4, there exists a function  $w(t) > 0$  which eventually satisfies

$$w(t) \geq Q_1(t) + \int_t^\infty \frac{w^2(s)}{A(s)} ds.$$

Set  $y(t) = \int_t^\infty (w^2(s)/A(s)) ds$ . Then  $y'(t) = -w^2(t)/A(t) < 0$  and

$$y'(s) \exp\left(2 \int_t^s \frac{Q_1(\theta)}{A(\theta)} d\theta\right) = -\frac{w^2(s)}{A(s)} \exp\left(2 \int_t^s \frac{Q_1(\theta)}{A(\theta)} d\theta\right). \quad (2.13)$$

Integrating (2.13) from  $t$  to  $T$ , then

$$\begin{aligned} y(T) \exp\left(2 \int_t^T \frac{Q_1(\theta)}{A(\theta)} d\theta\right) - y(t) - \int_t^T \frac{2y(s)Q_1(s)}{A(s)} \exp\left(2 \int_t^s \frac{Q_1(\theta)}{A(\theta)} d\theta\right) ds \\ = - \int_t^T \frac{w^2(s)}{A(s)} \exp\left(2 \int_t^s \frac{Q_1(\theta)}{A(\theta)} d\theta\right) ds. \end{aligned}$$

Therefore,

$$y(t) \geq \int_t^T \frac{\exp(2 \int_t^s (Q_1(\theta)/A(\theta)) d\theta)}{A(s)} (w^2(s) - 2y(s)Q_1(s)) ds.$$

Note that  $w(t) \geq Q_1(t) + y(t)$  and  $Q_1(t) > 0$ ,  $y(t) > 0$ . Hence

$$w^2(t) - 2Q_1(t)y(t) \geq Q_1^2(t) + y^2(t).$$

Thus we have

$$y(t) \geq \int_t^\infty \frac{Q_1^2(s)}{A(s)} \exp\left(2 \int_t^s \frac{Q_1(\theta)}{A(\theta)} d\theta\right) ds + \int_t^\infty \frac{y^2(s)}{A(s)} \exp\left(2 \int_t^s \frac{Q_1(\theta)}{A(\theta)} d\theta\right) ds,$$

which contradicts (2.12). The proof is complete.

We construct the following sequence  $\{Q_i(t)\}$  of functions for  $i = 1, 2, \dots, m$ .  $Q_i(t)$  is defined as

$$\begin{aligned} Q_1(t) &= \int_t^\infty A(s) \left\{ \bar{q}(s) + \left( \frac{A'(s)}{2A(s)} \right)^2 + \left( \frac{A'(s)}{2A(s)} \right) \right\} ds, \\ Q_2(t) &= \int_t^\infty \frac{Q_1^2(s)}{A(s)} \exp\left(2 \int_t^s \frac{Q_1(\theta)}{A(\theta)} d\theta\right) ds, \\ &\dots \\ Q_m(t) &= \int_t^\infty \frac{Q_{m-1}^2(s)}{A(s)} \exp\left(2 \int_t^s \frac{Q_{m-1}(\theta)}{A(\theta)} d\theta\right) ds. \end{aligned} \quad (2.14)$$

We will have the following theorem by using similar methods in the proof of Theorem 5.

THEOREM 6. If  $Q_i(t)$  in (2.14) exists, for  $i = 1, 2, 3, \dots, m_0 - 1$  but  $Q_{m_0}$  does not exist, where  $m_0 > 0$  is a positive integer, then every solution of (1.1) is oscillatory.

EXAMPLE 1. Consider the difference equation

$$\Delta_\tau^2 x(t) + \frac{1}{t} x(t - \sigma) = 0, \quad (2.15)$$

where  $\tau > 0$ ,  $\sigma > 0$ ,  $p(t) = \frac{1}{t}$ ,  $q(t) = \min_{t \leq s \leq t+2\tau} \{p(s)\} = \frac{1}{t+2\tau}$ , and

$$\sum_{i=1}^{\infty} \frac{1}{t+2\tau+i\tau} = \infty.$$

By Theorem 1, every solution of (2.15) is oscillatory, or

$$\int_t^\infty \frac{1}{(s+2\tau)\tau^2} ds = \infty.$$

By Theorem 3, every solution of (2.15) is oscillatory.

EXAMPLE 2. Consider the difference equation

$$\Delta_\tau^2 x(t) + \frac{\lambda}{(t-2\tau)^2} x(t - \sigma) = 0, \quad t > \tau, \quad (2.16)$$

where  $\tau > 0$ ,  $\sigma > 0$ ,  $\bar{q}(t) = \lambda/(t^2\tau^2)$ . Choose  $A(t) \equiv 1$  in Theorem 6,

$$\begin{aligned} Q_1(t) &= \int_t^\infty \frac{\lambda}{\tau^2 s^2} ds = \frac{\lambda}{\tau^2 t}, \\ Q_2(t) &= \int_t^\infty \frac{\lambda^2}{\tau^4 s^2} \exp\left(2 \int_t^s \frac{\lambda}{\tau^2 \theta} d\theta\right) ds \\ &= \frac{\lambda^2}{\tau^4 t^{\frac{2\lambda}{\tau^2}}} \int_t^\infty \frac{1}{s^{2(1-\frac{\lambda}{\tau^2})}} ds, \end{aligned}$$

where  $\lambda \geq \tau^2/2$ ,

$$\int_t^\infty \frac{1}{s^{2(1-\frac{\lambda}{\tau^2})}} ds = \infty.$$

By Theorem 6, every solution of (2.15) is oscillatory.

Remark 2. If (2.7), (2.10), and (2.12) do not hold, by using similar methods in the paper [9], we can obtain some *meticulous conditions* for the oscillation.



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